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OPTIMUM DESIGN FOR POTENTIAL FLOWS

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SUMMARY

Described in this paper is a methodology for solving a particular class of optimum design problems in Fluid Mechanics, namely optimum design problems for aerofoils when the corresponding fluid flow is potential. The methods described in this paper operate directly in the physical space, and take advantage of the variational formulation of the partial differential equation modelling the flow. The techniques of optimal control, optimization and the finite element method are used. Numerical examples are also given.

KEY WORDS Optimization Finite Elements Partial Differential Equations

1. INTRODUCTION

This paper deals with the numerical computation of optimal aerofoils. These problems have been studied in part by Miele,[29] Labrujere *et al.*,[36] and Glowinski and Pironneau.[35] From the mathematical point of view such problems are known as *optimal shape design* problems because one wishes to optimize a real-valued function of a solution of a Partial Differential Equation with respect to its domain of definition.

The most obvious criterion for a good aerofoil is a minimal drag for a given area and a given lift. However, the corresponding optimum design problem involves the full Navier–Stokes equation at high Reynolds number. As it is desirable to study inviscid potential the problem has been simplified by assuming that the drag is a monotonic function of the smoothness of the pressure distribution on the skin of the aerofoil, itself related to the size of the separated boundary layer near the trailing edge.

Once the criterion and the constraints on the shape are chosen, the equation can be discretized by the finite element method. Lions,[30] Cea *et al.*,[31] Pironneau,[32] Murat and Simon,[33] Marrocco and Pironneau,[7] and Rousselet[34] have shown that the techniques of optimal control can be used to solve the problem. In the present case a direct solution as in Miele[29] or an approximate solution by the method of local variations is not possible.

In this publication the method developed in Reference 7 will be applied to the present problem of aerodynamics; optimal shapes of nozzles and aerofoils will be computed with potential incompressible and compressible flows, with some incursion into the transonic regime.

In the first section the problem is defined; in the second the continuous problem is considered in the third discretization is effected during the finite element method of degree 1. Sections 4 and 5 are devoted to the motion of the triangulations and to the full statement of the algorithm. Sections 7 and 8 are dedicated to the lifting case and to non-differentiable criteria. Finally, the numerical results are presented in section 9. They show good agreement

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with exact solutions, when these are known and interesting features not in contradiction with the engineering know-how for the general case.

2. CHOICE OF AN AERODYNAMICAL CRITERION

In the aeronautical industry, numerical simulation may be used to estimate the performances of given shapes (lift and drag for example). They are of fundamental aid in the design of new aircraft. Engineers use their intuition and past experience to suggest a shape which will satisfy given criteria.

Here, optimal shapes for only one criterion are sought. In potential flows, the most important aerodynamical criteria are related to the pressure distribution on the skin of the aerofoil. It is well known that separated flows increase the resistance to the motion in a fluid (cf. Landau and Lipschitz[16]). It is desirable to delay the separations and wakes and these have to appear behind the aerofoil in such a way that the turbulent slip stream is as narrow as possible. But when there is a rapid decrease of the pressure along the side of the aerofoil in the direction of the flow, separated flows appear more easily. Therefore the shape of the aerofoil must be such that the pressure variation alongside of the aerofoil, where the pressure decreases, is the slowest and as continuous as possible.

Consideration can be given to the minimization of the gradient of the pressure coefficient C_p, but it is known that there are always large gradients of pressure in the neighbourhood of the leading edge. Thus it is desired to minimize the functional

$$\max_{x \in \text{aerofoil}} |C_p(x)|.$$

Let Σ_{ad} be the set of feasible shapes for the aerofoils. First, the $L^\infty(\Sigma_{ad})$ norm must be approximately by the L^m (Σ_{ad}) norm with m a large, even number. In doing so, a differentiable optimum design problem is obtained; a typical formulation is:

Find an aerofoil A *belonging to* Σ_{ad} *which minimizes*

$$E(A) = \int_A |C_p(s)|^m \, ds. \tag{1}$$

Then, using *non-smooth* optimization methods (see Lemaréchal[28]), also solve the following problem will also be solved.

Find an aerofoil A *belonging to* Σ_{ad} *which minimizes*

$$E(A) = \max_{x \in A} |C_p(x)|. \tag{2}$$

3. STATEMENT OF THE CONTINUOUS PROBLEM (INCOMPRESSIBLE CASE)

Consider, for simplicity, an incompressible potential flow in a nozzle (cf. Figure 1).

Let f and ϕ_0 be two given functions; m is an *even* number. The potential function ϕ is solution of

$$\left.\begin{aligned} \Delta\phi &= 0 \quad in \quad \Omega, \\ \phi &= \phi_0 \quad on \quad \Gamma_1, \\ \frac{\partial \phi}{\partial n} &= 0 \quad on \quad \Gamma. \end{aligned}\right\} \tag{3}$$

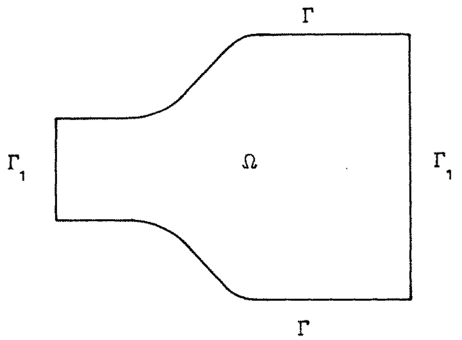


Figure 1. Typical nozzle configuration

Thus the potential function ϕ is a solution of the linear variational problem

$$\left.\begin{aligned}&\int_\Omega \nabla\phi\,.\,\nabla\omega\,\mathrm{d}x=0 \quad \forall\omega\in V_0(\Omega),\\ &\phi-\phi_0\in V_0(\Omega)\end{aligned}\right\} \tag{4}$$

with $V_0(\Omega)=\{\phi\in H^1(\Omega)\mid\phi=0 \text{ on } \Gamma_1\}$.

It is desired that the following optimum design problem is solved,

$$\min_{\Omega\in\Sigma_{ad}} E(\Omega) \tag{5}$$

where

$$E(\Omega)=\int_\Gamma (|\nabla\phi|^2-f)^m\,\mathrm{d}\gamma. \tag{6}$$

To solve such a problem, just discretize by the finite element method and then use optimization techniques which require a knowledge of the derivative of the criterion, E, with respect to the domain.

4. THE DISCRETE PROBLEM

4.1. Triangulation

The method of finite elements is particularly well suited to optimum design problems as triangles follow variations of the shape quite well.

As usual a triangulation $\mathcal{T}_h$ of the domain Ω_h is considered, which approximates Ω (cf. Zienkiewicz[17]). For simplicity it is supposed that Ω is a polygon, so that $\Omega_h=\Omega$. Then $\mathcal{T}_h$ is a set of triangles T such that

$$\left.\begin{aligned}&\bigcup_{T\in\mathcal{T}_h} T=\bar\Omega,\\ &\text{if } T_1 \text{ and } T_2 \text{ belong to } \mathcal{T}_h \text{ and if } T_1\neq T_2, \text{ we have } T_1\cap T_2=\emptyset,\\ &\text{or } T_1 \text{ and } T_2 \text{ have a common side,}\\ &\text{or } T_1 \text{ and } T_2 \text{ have a common vertex.}\end{aligned}\right\} \tag{7}$$

4.2. *Space approximation*

Let us define $V_{0h}=\{v_h\in \boldsymbol{C}^0(\bar{\Omega})/v_{h|T}\in P_1(T),\ v_{h|\Gamma_{1h}}=0\forall T\in \boldsymbol{T}_h\}$ where $P_1(T)$ is the set of polynomials defined on T, whose degree is less than or equal to one; V_{0h} is an approximation of V_0. A function v_h of V_{0h} is completely determined by the values it takes on the set of nodes of $\boldsymbol{T}_h$ which are not on Γ_{1h}, the approximation of Γ_1; this set has $N(h)$ nodes. Define the functions w_i by

$$\left.\begin{aligned} &w_j\in V_{0h}\\ &w_j(x_i)=\delta_{ij}\ \textit{for}\ i=1,\ldots,N(h).\end{aligned}\right\}\qquad (8)$$

Then $\{w_j\}_{j=1}^{N(h)}$ is a basis of V_{0h}, and the discrete potential function ϕ_h may be explicited as follows:

$$\left.\begin{aligned} &\phi_h(x)=\sum_{i=1}^{N(h)}\phi_i w_i(x),\\ &\textit{with}\ \phi_i=\phi_h(x_i).\end{aligned}\right\}\qquad (9)$$

4.3. *Discretization of the problem*

To obtain an approximation of the derivative $E'(\Omega)$ of $E(\Omega)$ one can discretize directly. An operator with bad computational properties is usually obtained. A 'safe' approach is to discretize $E(\Omega)$, first, and then to calculate its derivative *exactly*. A discrete optimum design, is therefore desirable.

The discrete potential function ϕ_h is a solution of

$$\left.\begin{aligned} &\int_{\Omega_h}\boldsymbol{\nabla}\phi_h\,.\,\boldsymbol{\nabla}\omega_h\,\mathrm{d}x=0\quad \forall\omega_h\in V_{0h},\\ &\phi_h-\phi_{0h}\in V_{0h},\end{aligned}\right\}\qquad (10)$$

where ϕ_{0h} is an approximation of ϕ_0.

Define, therefore,

$$E_h(\boldsymbol{T}_h)=\int_{A_h}(|\boldsymbol{\nabla}\phi_h|^2-f)^m\,\mathrm{d}x \qquad (11)$$

where A_h in (11) is a subdomain of Ω_h 'approximating' the unknown part of the boundary as shown on Figure 2.

The set A_h allows the transformation of (6) into a surface integral. It is desired to find the co-ordinates of the nodes of $\boldsymbol{T}_h$ such that $E_h(\boldsymbol{T}_h)$ is a minimum. Thus if χ is the set of feasible locations of the nodes $\{x_k\}$, $k=1,\ldots,N$ for the triangulation, then the problem is:

$$\min_{\boldsymbol{T}_h\in\chi}E_h(\boldsymbol{T}_h). \qquad (12)$$

This problem is a constrained minimization problem in $R^{2\times N}$.

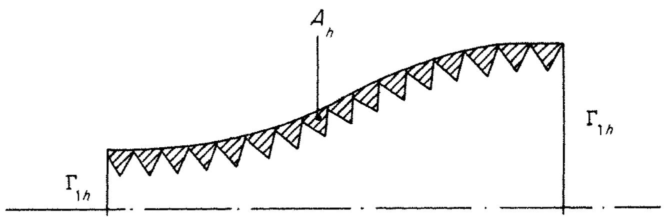


Figure 2. Representation of a half-nozzle

In order to use gradient methods compute

$$\left.\begin{aligned}\partial_{\alpha_{j,i}}E_h(T_h)&=\lim_{|\alpha_{j,i}|\to 0}[E_h(T'_h)-E_h(T_h)]/\alpha_{j,i}\\ i=1,2,\quad j&=1,\dots,N,\end{aligned}\right\}\tag{13}$$

where T'_h is the triangulation obtained from T_h by moving the jth node in position $x_j+\boldsymbol{\alpha}_j$ where $\boldsymbol{\alpha}_j=\{\alpha_{j,1},\alpha_{j,2}\}$. Therefore, if f is piecewise constant on T_h, we have

$$\left.\begin{aligned}\delta E_h&=E_h(T'_h)-E_h(T_h)\\ \delta E_h&=\int_{A'_h}(|\nabla\phi'_h|^2-f)^m\,dx-\int_{A_h}(|\nabla\phi_h|^2-f)^m\,dx,\end{aligned}\right\}\tag{14}$$

where ϕ'_h is computed on T'_h by (10) and where A'_h is the analogue of A_h for T'_h.

If C and C' are two sets of $\mathbb{R}^2$, denote

$$C\setminus C'=C-(C'\cap C),$$
$$C'\setminus C=C'-(C'\cap C).$$

Then (14) can be rewritten as

$$\left.\begin{aligned}\delta E_h=&\int_{A'_h\cap A_h}[(|\nabla\phi'_h|^2-f)^m-(|\nabla\phi_h|^2-f)^m]\,dx\\ &+\int_{A'_h\setminus A_h}(|\nabla\phi'_h|^2-f)^m\,dx-\int_{A_h\setminus A'_h}(|\nabla\phi_h|^2-f)^m\,dx.\end{aligned}\right\}\tag{15}$$

Define

$$\left.\begin{aligned}&\delta\phi_i=\phi'_i-\phi_i\ on\ T_h\cap T'_h,\\ &\delta\omega_i=\omega'_i-\omega_i,\ i=1,\dots,N(h),\\ &\delta\phi_h=\phi'_h-\phi_h=\sum_{i=1}^{N(h)}\phi'_i\omega'_i-\sum_{i=1}^{N(h)}\phi_i\omega_i,\\ &\delta\phi_h=\sum_{i=1}^{N(h)}(\phi_i+\delta\phi_i)(\omega_i+\delta\omega_i)-\sum_{i=1}^{N(h)}\phi_i\omega_i.\end{aligned}\right\}\tag{16}$$

Lemma 1. *If the triangulation T'_h is obtained from T_h by moving the* jth *node from x_j to $x_j+\alpha_j$, then*

$$\left.\begin{aligned}&\delta\omega_i(x)=-\omega_j(x)(\nabla\omega_i\cdot\boldsymbol{\alpha}_j)+O(|\boldsymbol{\alpha}_j|)\\ &\forall x\in T_k\cap T'_k,\quad \forall i,k\ such\ that\ x_i,x_j\in T_k;\end{aligned}\right\}$$

if $x_j\notin T_k$ then $\delta\omega_i(x)=0$.

Proof: (see Marrocco and Pironneau.[7]) (See Figure 3.)

Lemma 2. *Let f be a function piecewise constant on $\Omega=\bigcup_j T_j$, then if x_j goes to $x_j+\boldsymbol{\alpha}_j$, we have*

$$\left.\sum_k\int_{T'_k\setminus T_k}f\,dx-\int_{T_k\setminus T'_k}f\,dx=\int_\Omega f(\nabla\omega_j\cdot\boldsymbol{\alpha}_j)\,dx+O(|\boldsymbol{\alpha}_j|).\right\}$$

Proof: (see Marrocco and Pirroneau.[7])

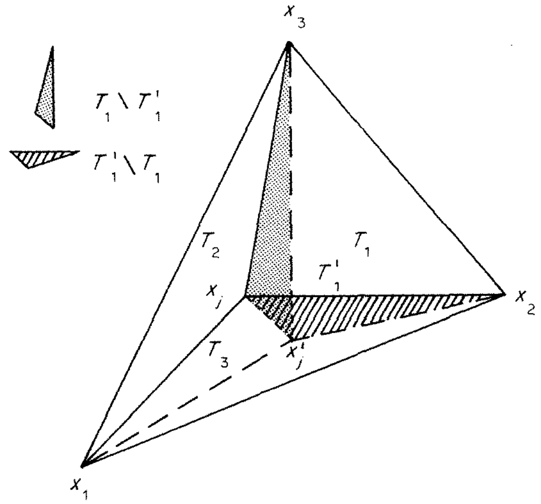


Figure 3. Transformation of the triangulation when the node x_j is moved. $T_1' T_1$ and $T\setminus T_1'$ are shown

Let $\boldsymbol{\alpha}_j(x)$ be the function which is constant on T_k such that

$$\left.\begin{array}{lll}\tilde{\boldsymbol{\alpha}}_j(x)=\boldsymbol{\alpha}_j & \text{if} & x\in T_k \text{ and if } x_j\in T_k\\ \boldsymbol{\alpha}_j(x)=\mathbf{0} & \text{if} & x\in T_k \text{ and if } x_j\notin T_k.\end{array}\right\} \tag{17}$$

Now, we can compute the variation $\delta\phi_h$ of ϕ_h.

$$\delta\phi_h=\sum_{i=1}^{N(h)}\delta\phi_i\omega_i+\sum_{i=1}^{N(h)}\phi_i\,\delta\omega_i+O(|\boldsymbol{\alpha}_j|).$$

Define

$$\delta\tilde{\phi}=\sum_{i=1}^{N(h)}\delta\phi_i\omega_i;$$

using the above two lemmas, we find

$$\left.\begin{aligned}\delta E_h=2m&\int_{A_h}[|\nabla\phi_h|^2-f]^{m-1}(\nabla\phi_h\,.\,\nabla\delta\tilde{\phi})\,\mathrm{d}x\\ -2m&\int_{A_h}[|\nabla\phi_h|^2-f]^{m-1}(\nabla\phi_h\,.\,\nabla\omega_j(\nabla\phi_h\,.\,\boldsymbol{\alpha}_j))\,\mathrm{d}x\\ +&\int_{A_h}[|\nabla\phi_h|^2-f]^m(\nabla\omega_j\,.\,\boldsymbol{\alpha}_j)\,\mathrm{d}x+O(|\boldsymbol{\alpha}_j|).\end{aligned}\right\} \tag{18}$$

So to evaluate the first term in (18) we must introduce the solution of the adjoint equation below:

$$\left.\begin{array}{l}\Pi_h\in V_{0h}\\ \displaystyle\int_{\Omega_h}\nabla\Pi_h\,.\,\nabla\omega_h\,\mathrm{d}x=2m\int_{A_h}(\nabla\phi_h\,.\,\nabla\omega_h)(|\nabla\phi_h|^2-f)^{m-1}\,\mathrm{d}x\forall\omega_h\in V_{0h}.\end{array}\right\} \tag{19}$$

Using (10) and lemma 1, it is found that

$$\left.\begin{aligned}\delta_{\alpha_j}E_h(T_h)=\int_{\Omega_h}(\nabla\Pi_h\cdot\nabla\omega_j)(\nabla\phi_h\cdot D_j)\,\mathrm{d}x+\int_{\Omega_h}(\nabla\phi_h\cdot\nabla\omega_j)(\nabla\Pi_h\cdot\mathbf{D}_j)\,\mathrm{d}x\\ -\int_{\Omega_h}(\nabla\phi_h\cdot\nabla\omega_h)(\nabla\omega_j\cdot\mathbf{D}_j)\,\mathrm{d}x-2m\int_{A_h}(|\nabla\phi_h|^2-f)^{m-1}(\nabla\phi_h\cdot\nabla\omega_j(\nabla\phi_h\cdot\mathbf{D}_j))\,\mathrm{d}x\\ +\int_{A_h}(|\nabla\phi_h|^2-f)^m(\nabla\omega_j\mathbf{D}_j)\,\mathrm{d}x\end{aligned}\right\}\quad(20)$$

where $\mathbf{D}_j=\boldsymbol{\alpha}_j\setminus|\boldsymbol{\alpha}_j|$.

Equation (4.6) can now be solved quite readily by any optimization technique on making use of the differential of the cost criterion.

5. MOTION OF THE NODES

It is desired to find the shape of a nozzle (or of an aerofoil). In this case χ is the set of co-ordinates of the nodes of all proper triangulations of Ω, with the same topological properties (same number of triangles, same number of nodes . . .).

The nodes of the triangulation can be moved using various methods.

5.1. *First method*

To remove the constraints on $\{x_k\}_1^N$ it is useful to assume that all interior nodes are constructed from the nodes of the unknown boundary using a set of continuous mappings; therefore assume that there exists a function X_k, such that

$$x_k=X_k(x_1,\ldots,x_s),\ k=s+1,\ldots,N. \quad (21)$$

Such nodes will be called *associated moving nodes.*

It will be seen later that it is not necessary to know the functions X_k explicitly; in fact, only the derivatives $\partial X_k/\partial x_j$ are required.

Furthermore, in order to avoid the nodes of the unknown boundary crossing each other, they can be constrained to remain on prescribed curves. It should be kept in mind that the solution may depend upon these curves. So

$$x_j=X_j(t_j)\qquad j=1,\ldots,s,\ t_j\in\mathbb{R}. \quad (22)$$

These nodes are called the *principal moving nodes.*

Now (12) is an unconstrained problem in $t\in\mathbb{R}^s$ (with respect to the t_j's) and from (20) and the theorem on the derivation of composite functions we have

$$\frac{\partial E_h}{\partial t_j}=\left[\sum_{k=s+1}^{N}\partial_{\alpha_k}E_h\alpha_k^j\right]+\partial_{\alpha_j}E_h\frac{\partial X_j}{\partial t_j}\quad \textit{with}\quad \alpha_k^j=\partial_{x_j}\frac{\partial X_j}{\partial t_j},\qquad k=s+1,\ldots,N(h). \quad (23)$$

5.2. *Second method*

When a gradient method with optimal step size is used, every node can be a principal moving node, but a restriction must be imposed on the step size $\lambda_{\max}$, with $\lambda_{\max}$ depending upon the triangulation.

5.3. Remark

To avoid oscillations on Γ only half of the nodes on the boundary Γ are moved independently. For example, if (21) is replaced by

$$\left.\begin{aligned} &x_j = k_j(t_j), \quad j=2k-1, \quad k=1,\ldots,s/2 \\ &x_j = \alpha x_{j-1} + (1-\alpha)x_{j+1}, \quad j=2k, \quad k=1,\ldots,s/2 \end{aligned}\right\} \tag{21'}$$

s has to be even.

6. AN OPTIMIZATION ALGORITHM

An optimal design algorithm belonging to the family of the gradient methods with a fixed step size is described by:

0. Choose t^0 and λ.
   Set $i=0$.
   Compute the triangulation from (21) and (22).
1. Compute ϕ_h and Π_h from (10) and (19).
2. Compute $\dfrac{\partial x_i}{\partial x_j}, \dfrac{\partial x_j}{\partial t_j}$ for $j=1,\ldots,s$, $i=1,\ldots,N$
   Compute $\partial_{\alpha_i} E_h$ from (20).
   Compute $\dfrac{\partial E_h}{\partial t_j}$ from (23).
3. Set $h_j = -\dfrac{\partial E_h}{\partial t_j}$, $j=1,\ldots,s$.
   Set $t_j^{i+1} = t_j^i + \lambda h_j$.
   Set $i=i+1$.
   Compute the new triangulation and go back to 1.

7. OPTIMUM DESIGN FOR LIFTING FLOWS

7.1. The continuous problem

Consider a lifting potential flow around a profile (see Figure 4).

7.1.1. *Mathematical formulation of the physical problem.* The state equations hold for either incompressible or compressible fluids

$$\left.\begin{aligned} &\nabla\cdot\rho\nabla\phi = 0 \quad \text{in} \quad \Omega, \\ &\rho = (1-k\,|\nabla\phi|^2)^{1/\gamma-1} \quad \text{in} \quad \Omega, \end{aligned}\right\} \tag{24}$$

$$\frac{\partial\phi}{\partial n} = \mathbf{v}_\infty\cdot\mathbf{n} \quad \text{on} \quad \Gamma_\infty, \tag{25}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad \Gamma, \tag{26}$$

$$[\phi]_\Sigma = \alpha, \tag{27}$$

$$|\nabla\phi(TE^+)|^2 - |\nabla\phi(TE^-)|^2 = 0 \tag{28}$$

$$\phi = 0 \quad \text{at} \quad TE^+. \tag{29}$$

See Figure 5 for the definition of TE^+ and TE^-; we have: $k = \dfrac{\gamma-1}{\gamma+1}\dfrac{1}{c^*}$ (resp. $k=0$) for

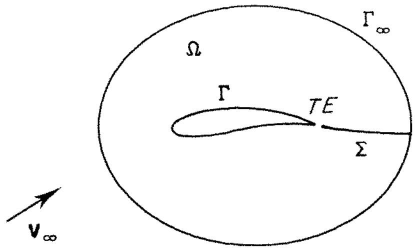


Figure 4. Typical profile configuration

compressible (resp. incompressible) fluids, where c_* is the critical velocity and γ the ratio of specific heats ($\gamma = 1{\cdot}4$ in air).

$\mathbf{v}_\infty$ is the flow velocity at infinity, Σ is a cutting line, $[\phi]_\Sigma$ is the jump of ϕ across Σ, relation (28) is the Joukowsky condition at the trailing edge. If the fluid is incompressible (24) reduces to $\Delta\phi = 0$.

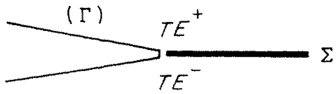


Figure 5. Trailing edge

This criterion for *incompressible* fluids is

$$E(\Gamma, \phi) = \int_\Gamma (|\nabla\phi|^2 - f)^m \,\mathrm{d}x \tag{30a}$$

and for *compressible* fluids is

$$E(\Gamma, \phi) = \int_\Gamma (\rho^\gamma - f)^m \,\mathrm{d}x. \tag{30b}$$

7.1.2. *Solution of the state equation.* To solve the state equation ϕ is written as $\phi_1 + \phi_2$ (see for example References 18 and 22); ϕ_2 is a continuous function and ϕ_1 is a discontinuous one which satisfies (24), (26), (29), and also

$$\frac{\partial\phi_1}{\partial n} = 0 \text{ on } \Gamma_\infty \tag{31}$$

$$[\phi_1]_\Sigma = 1. \tag{32}$$

Function ϕ_2 satisfies (24), (25), (26) and (29) where ρ is calculated using the function $\phi = \alpha\phi_1 + \alpha_2$ which satisfies the state equations (24), (25), (26), (27), (29).

So, define

$$J = |\nabla\phi(TE^+)|^2 - |\nabla\phi(TE^-)|^2.$$

To solve $J = 0$ a one-dimensional Newton method (or a secant method, alternatively) is used.

7.1.3. *Calculation of the gradient.* The lift, α, is given. To satisfy the Joukowsky condition, the aerofoil must be in incidence.

Let θ be the angle of attack, so the potential function ϕ depends on the boundary Γ and on the angle θ. So the notation $J(\Gamma, \theta)$ and $E(\Gamma, \theta)$ can be used. It is desired to calculate the derivatives of E and J to solve (1) by optimization methods using the gradient of the criterion.

We have formally:

$$J(\Gamma,\theta)=0 \Rightarrow \theta=f(\Gamma), \quad \text{so} \quad E(\Gamma,\theta)=E(\Gamma,f(\Gamma))=\tilde{E}(\Gamma).$$

Consider $\delta\tilde{E}=\tilde{E}(\Gamma+\delta\Gamma)-\tilde{E}(\Gamma)$; we have

$$\delta\tilde{E}=\frac{\partial E}{\partial\Gamma}\,\delta T+\frac{\partial E}{\partial\theta}\,\delta\theta,$$

$$\delta\tilde{E}=\left(\frac{\partial E}{\partial\Gamma}+\frac{\partial E}{\partial\theta}\frac{\partial f}{\partial\Gamma}\right)\delta\Gamma,$$

$$\frac{\partial f}{\partial\Gamma}=-\left[\frac{\partial J}{\partial\theta}\right]^{-1}\frac{\partial J}{\partial\Gamma},$$

implying

$$\delta\tilde{E}=\left[\frac{\partial E}{\partial\Gamma}-\frac{\partial E}{\partial\theta}\left[\frac{\partial J}{\partial\theta}\right]^{-1}\frac{\partial J}{\partial\Gamma}\right]\delta\Gamma.$$

7.2. The discrete problem

It is desired to minimize

$$E_h=\int_{A_h}(|\nabla\phi_h|^2-f_h)^m\,dx,$$

where ϕ_h satisfies

$$\left.\begin{aligned}&\phi_h\in V_{1h},\\&\int_\Omega \nabla\phi_h\cdot\nabla\omega_h\,dx=\int_{\Gamma_\infty}\mathbf{v}_\infty\cdot\mathbf{n}\omega_h\,d\gamma \qquad \forall\omega_h\in V_{0h},\\&[\phi_h]_\Sigma=\alpha\end{aligned}\right\}$$

$$J_h=|\nabla\phi_h(T^+)|^2-|\nabla\phi_h(T^-)|^2=0,$$

with

T^+ and T^- triangles near the trailing edge (see Figure 6)

$V_{1h}=\{\phi_h\in C(\Omega_h\setminus\Sigma_h)\mid \phi_{h|_T}\in P_1,\ \phi_h(TE^+)=0,\ \phi_h$ satisfies $J_h=0\}$,

$V_{0h}=\{\omega_h\in C(\Omega_h)\mid \omega_{h|_T}\in P_1,\ \omega_h(TE)=0\}$

To compute $\partial_{\alpha_i}E_h$ and $\partial_{\theta_h}E_h$ we need an adjoint state solution Π_1; Π_1 satisfies

$$\left.\begin{aligned}&\Pi_1\in V_{0h},\\&\int_{\Omega_h}(\nabla\Pi_1\cdot\nabla\omega_h)\,dx=2m\int_{A_h}(|\nabla\phi_h|^2-f_h)^{m-1}(\nabla\phi_h-\nabla\omega_h)\,dx,\end{aligned}\right\}$$

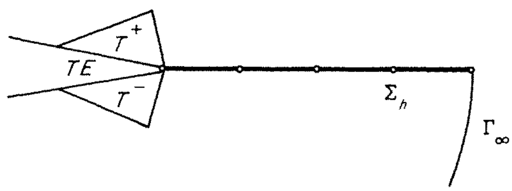


Figure 6. Discretization near the trailing edge

then

$$\left.\begin{aligned}\partial_{\alpha_j}E_h(T_h,\theta_h)=&\int_{\Omega_h}(\nabla\Pi_1\cdot\nabla\omega_j)(\nabla\phi_h\cdot\mathbf{D}_j)\,\mathrm{d}x\\&+\int_{\Omega_h}(\nabla\phi_h\cdot\nabla\omega_j)(\nabla\Pi_1\cdot\mathbf{D}_j)\,\mathrm{d}x-\int_{\Omega_h}(\nabla\phi_h\cdot\nabla\Pi_1)(\nabla\omega_j\cdot\mathbf{D}_j)\,\mathrm{d}x\\&-2m\int_{A_h}[|\nabla\phi_h|^2-f]m^{-1}(\nabla\phi_h\cdot\nabla\omega_j)(\nabla\phi_h\cdot\mathbf{D}_j)\,\mathrm{d}x\\&+\int_{A_h}(|\nabla\phi_h|^2-f)^m(\nabla\omega_j\cdot\mathbf{D}_j)\,\mathrm{d}x\end{aligned}\right\}$$

and

$$\partial_{\theta_h}E_h(T_h,\theta_h)=\int_{\Gamma_\infty}\left(\mathbf{v}_\infty\cdot\frac{\partial}{\partial\theta}\mathbf{n}\right)\Pi_1\,\mathrm{d}\gamma.$$

For J_h an adjoint state function Π_2 is used; Π_2 satisfies

$$\left.\begin{aligned}&\Pi_2\in V_{0h}\\&\int_{\Omega_h}(\nabla\Pi_2\cdot\nabla\omega_h)\,\mathrm{d}x=2(\nabla\phi_h(T^+)\cdot\nabla\omega_h(T^+)-\nabla\phi_h(T^-)\nabla\omega_h(T^-)),\end{aligned}\right\}$$

then

$$\left.\begin{aligned}\partial_{\alpha_j}J_h(T_h,\theta_h)=&-2(\nabla\phi_h(T^+)\cdot\nabla\omega_j(T^+))(\nabla\phi_h(T^+)\cdot\mathbf{D}_j)\\&+2(\nabla\phi_h(T^-)\cdot\nabla\omega_j(T^-))(\nabla\phi_h(T^-)\cdot\mathbf{D}_j)\\&+\int_{\Omega_h}(\nabla\phi_h\cdot\nabla\omega_h)(\nabla\Pi_2\cdot\mathbf{D}_j)\,\mathrm{d}x\\&+\int_{\Omega_h}(\nabla\Pi_2\cdot\nabla\omega_j)(\nabla\phi_h\cdot\mathbf{D}_j)\,\mathrm{d}x\\&-\int_{\Omega_h}(\nabla\phi_h\cdot\nabla\Pi_2)(\nabla\omega_j\cdot\mathbf{D}_j)\,\mathrm{d}x,\end{aligned}\right\}$$

and

$$\partial_{\theta_h}J_h(T_h,u_h)=\int_{\Gamma_\infty}\left(\mathbf{v}_\infty\cdot\frac{\partial\mathbf{n}}{\partial\theta}\right)\Pi_2\,\mathrm{d}\gamma$$

8. SOME GENERALITIES ABOUT OPTIMUM DESIGN FOR NON-SMOOTH CRITERIA

As indicated in section 2 we are concerned with the non-differentiable criterion

$$E(A)=\max_{x\in A}|C_p(x)|$$

$E(A)$ is a function whose gradient is not continuous, but it is differentiable almost everywhere. To minimize such functions, the oldest methods used are relaxation-type and cutting plane. In these methods, the objective function is not monotonically decreasing from iteration to iteration. More recently, a new class of methods has been introduced, which retains the descent property, in which the direction is computed by projecting the origin onto a polyhedron by a set of gradients. An algorithm developped by Lemaréchal *et al.*[28] will be used. It uses another method in which the computation of the direction is slightly more sophisticated. In Reference 28, a subroutine is given which can treat linearly constrained problems, such that

$$\left.\begin{array}{ll} \min\limits_{x} E(x) & \\ Ax=b & x\in\mathbb{R}^n \\ a_1\leqslant x\leqslant a_2 & \end{array}\right\}$$

where E is locally Lipschitz continuous. This optimization subroutine is used for differentiable or non-differentiable optimum design problems. It is very efficient for solving the optimum design problems discussed in this paper.

9. NUMERICAL RESULTS

The method for incompressible and compressible potential flows was tested. In most cases the discretized criteria

$$E_h^1=\sum_{T\in A_h}\frac{S(T)}{h(T)}(|\nabla\phi_h|^2-f)^2 \quad \text{for incompressible flows}$$

or

$$E_h^2=\sum_{T\in A_h}\frac{S(T)}{h(T)}(\rho_h^\gamma-f)^m \quad \text{for compressible flows}$$

are used. $S(T)$ is the area of the triangle T belonging to A_h (see Figure 2). $h(T)$ is the distance between the frontier Γ_h and the node belonging to T which is not on Γ_h.

Two types of problems are solved. First inverse problems are studied. The specification of a desired pressure distribution is given and the corresponding shape has to be calculated. So f is a given function and the criterion has to be equal to zero at the end of the computation.

Direct design problems are also studied. Geometries that are in some sense optimized for a specific condition have to be designed. For example, the area of the profile has to be equal to some given value. In that case the function f is equal to the pressure at infinity.

9.1. Optimum design for nozzles

Triangulations with 90 nodes and 140 triangles are used. Computation always begins with a rectangle. This rectangle corresponds to a half nozzle. The middle of the nozzle is fixed for symmetry reasons. The other nodes can move on vertical lines, with the first method. The functions X_k are similarities on these lines.

We have an inverse problem for incompressible flows. The state equation is solved by the Cholesky method. Figure 7(a) represents the initial triangulation. Figure 7(b) corresponds to the desired shape of the nozzle with its triangulation. In Figure 7(c) the evolution of Γ_h during a few iterations can be seen. In Figure 7(d) the computed solution can be seen. Note that the final curve Γ_h is the same as the desired one, but the area of the final nozzle is

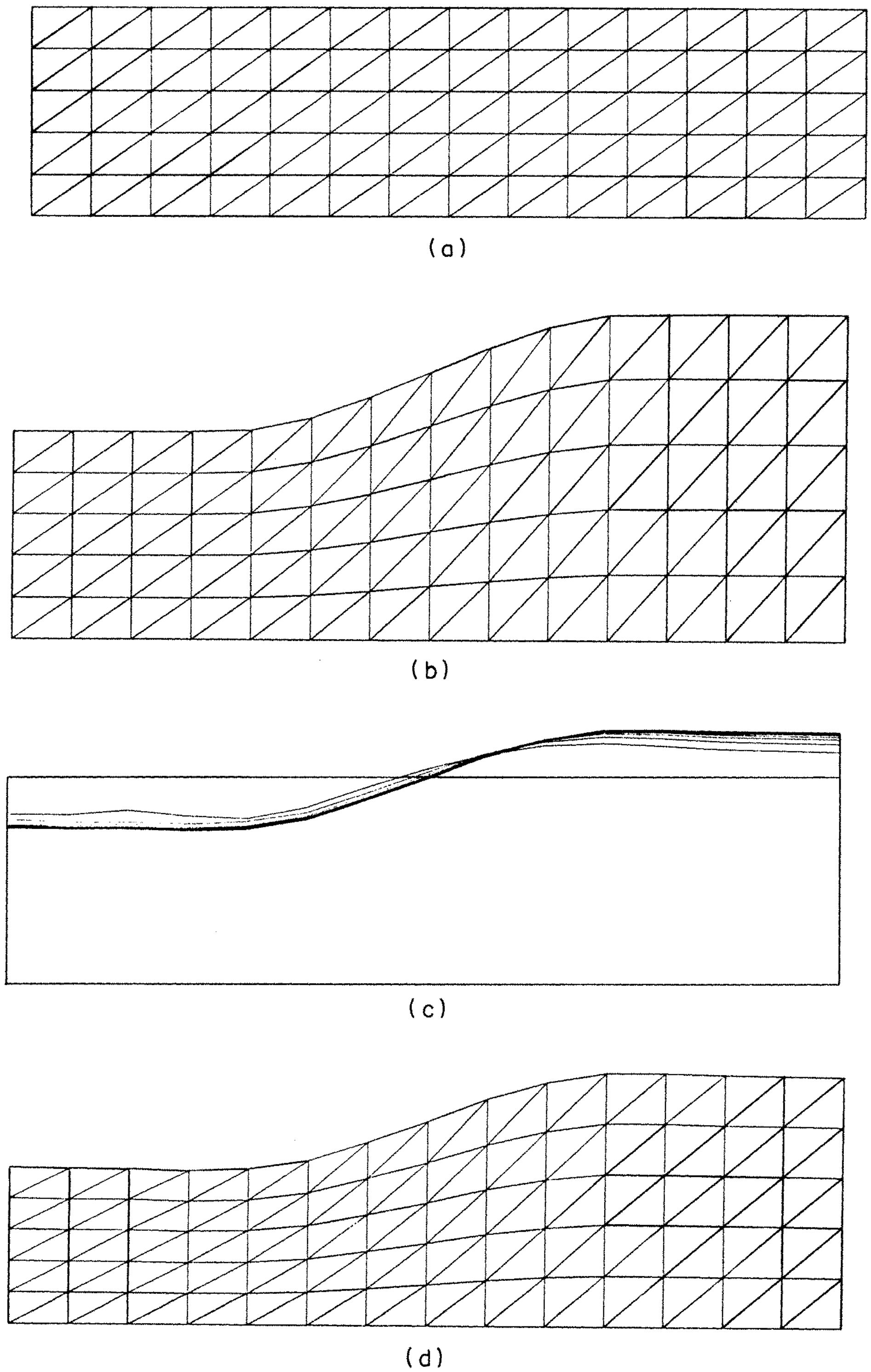
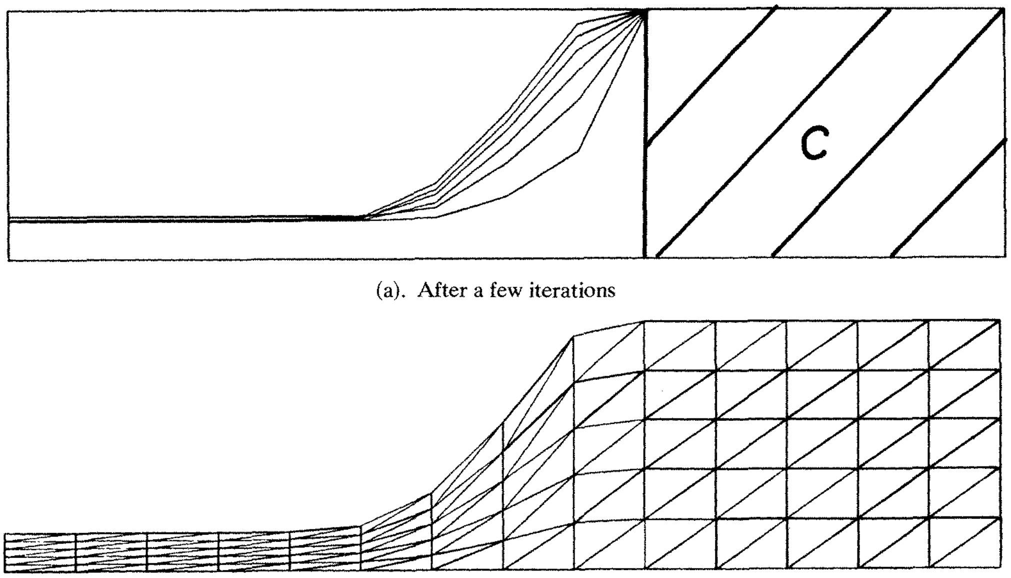


Figure 7



(a). After a few iterations

(b). Final result

Figure 8

smaller than the desired one. The initial value of E_h^1 is 0·14. After 40 iterations the value is $0{\cdot}7\times10^{-4}$. This computation takes 12s on an IBM 3033.

In Figure 8 a nozzle is optimized for incompressible fluids. In portion C of the nozzle a given flow is desired. So the criterion

$$E=\int_C (\nabla\phi_h-\mathbf{f})^2\,dx$$

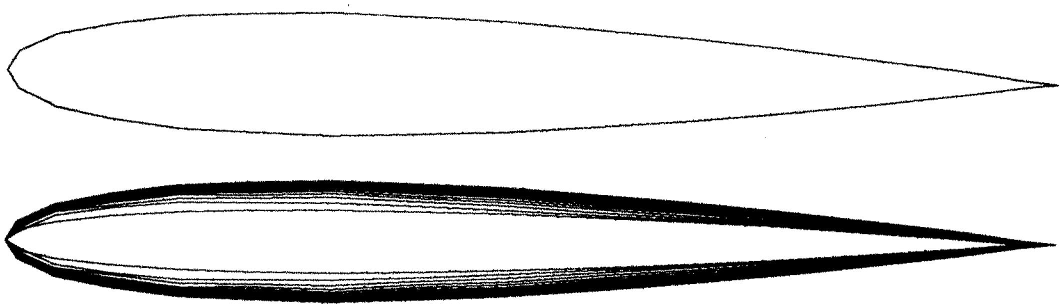
is minimized, where $\mathbf{f}$ is the desired velocity vector. Thirty iterations are necessary. The initial value of E is $0{\cdot}4\times10^{-1}$; the final value is $0{\cdot}8\times10^{-4}$. It takes 9 s on the IBM 3033.

9.2. Optimum design for profiles

To solve optimum design around profiles triangulations with 600 vertices and 1080 triangles are used. It is supposed that the flow is uniform at infinity.

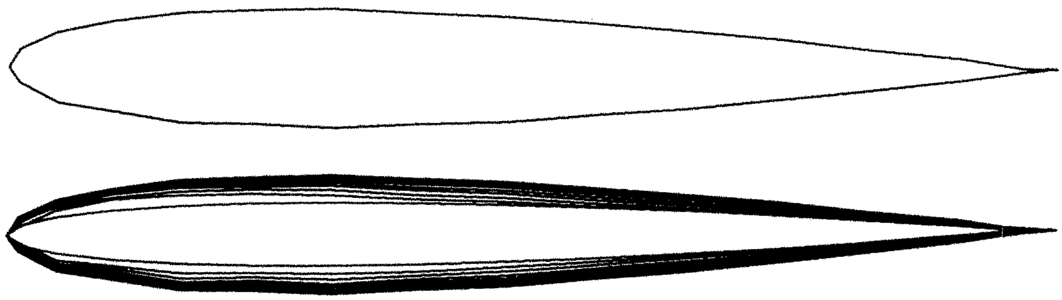
In Figure 9 this is an inverse problem for a non-lifting case; the flow is incompressible. The gradient method is used with optimal step size bounded with λ_{max}. The initial profile has a 6 per cent thickness. The desired profile is the NACAOO12. To avoid oscillations it is important to move the nodes on Γ_h by using formula (21′). To have good accuracy at the trailing edge a greater number of iterations are needed than for nozzles, but a good approximation of the desired profile with the same number of iterations has already been obtained. Here we have the comparison between 40 and 70 iterations.

In Figure 10 an inverse problem for compressible flows is solved. Optimal control[21] is used to calculate the state equation. The initial profile is the NACAOO12; the desired profile is the Korn one. This is a lifting compressible case with a differentiable criterion. Iteration is carried out on the Section 8 optimization algorithm. We note that between the second and third iterations only the region near the trailing edge has moved $\alpha=0.03$. It takes one hour on the IBM 3033. The method with optimal step size bounded with λ_{max} is slower than this one.



40 iterations

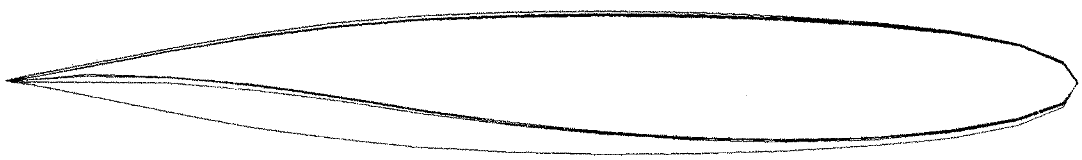
(a)



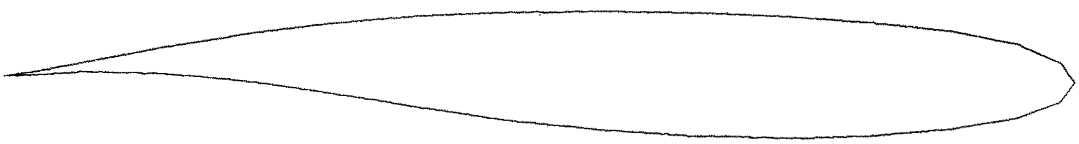
70 iterations

(b)

Figure 9



(a). After a few iterations



(b). Final profile

Figure 10

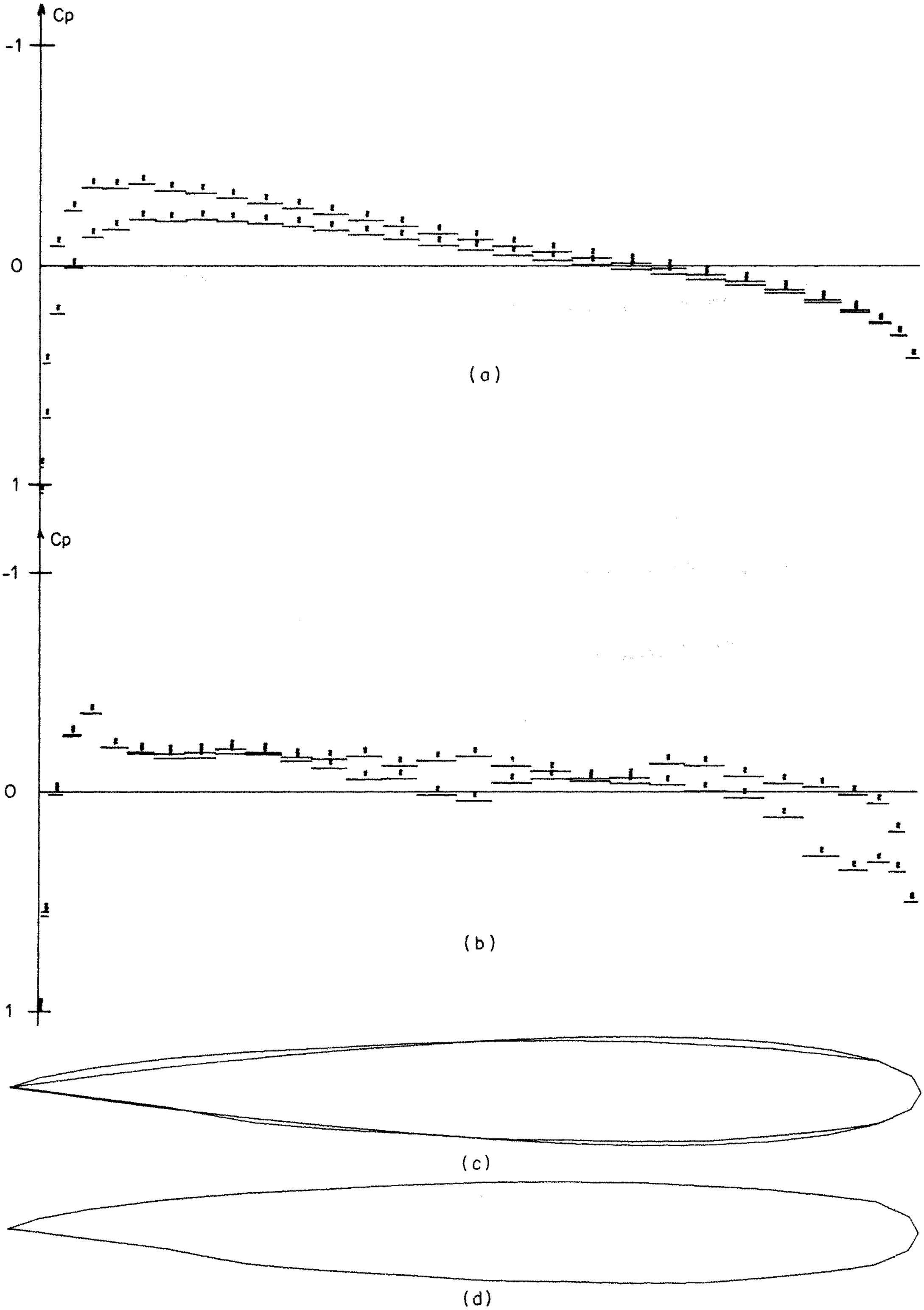


Figure 11

Figure 11 is an optimization problem for subsonic compressible flows. This is a lifting case, $\alpha = 0{\cdot}02$, $|\mathbf{V}_{\text{inf}}| = 0{\cdot}5$. It is desired to optimize the NACAOO12. The desired profile must have the same area as the initial one. The 9 nodes at the leading edge are fixed to avoid obtaining an angle. The trailing edge is fixed. The algorithm is used; the vertices on the profile satisfy (21′). The initial criterion was 1, the final one 0·53. It takes one hour on the IBM 3033. In Figure 11(a) the pressure on NACAOO12 and in Figure 11(b) the pressure on the optimized profile can be seen.

An optimum problem for transonic flows has also been studied. The method works but is too expensive. The state and costate equations have to be solved exactly. To do this the optimal control method with upwinding (there is no parameter to adjust) is used. But this is an iterative method. The required number of iterations increases greatly with the Mach number. For subsonic flows, about 10 iterations are needed, for transonic flows it can be 100 iterations. And we have to iterate on these problems. So only inverse problems are tested, not very spectacular results. The method used to solve the transonic potential flows has to be improved.

Non-differentiable criteria are also used for inverse problems; this works very well. The results are similar to the corresponding differentiable criteria.

10. CONCLUSIONS

The method discussed in this paper to solve optimum design problems was introduced by Marrocco and Pironneau[7] for the design of electromagnets. Here it has been seen that it is also well suited to optimum design problems in fluid mechanics.

As there is no parameter to adjust it can be used quite easily by design engineers in industrial situations.

Many variations of the above problems can be considered (other types of constraints, of cost criterion, etc. . .).

A next step will be to study three-dimensional problems.

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